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Mathematics Entry-Level Competencies

Incoming college level students are expected to bring hands-on skills in computation and algebraic manipulation, as well as conceptual knowledge rooted in a deep and solid understanding of numbers and basic geometry. Incoming students are expected to know basic mathematical concepts in computation, algebra and geometry. These are described in some detail in following sections.

In addition, incoming students need to have a comfort level with mathematics so that they approach problems by investigating their nature, asking questions, and revising approaches as they reflect on them. It is as important to understand why a solution works as it is to know how one reaches a solution.

Mastery of mathematics at all levels should include the following characteristics:

1. Thinking conceptually and not just procedurally about mathematics. Mathematics is a way of understanding, a thinking process, and not a collection of detached procedures to be learned and applied separately.
2. Using logical reasoning and common sense to work on problems in order to find solutions. Successful students can explain their processes and can check their solutions to see whether their findings make sense.
3. Using experimental thinking and a willingness to investigate the steps used to reach a solution, and recognizing that there are often multiple approaches to solving a problem.
4. Taking risks and accepting that a first or second attempt may result in a wrong answer, but that each attempt is an opportunity to try new approaches toward solving the problem.
5. Understanding that formulas and algorithms in computation, while important and crucial, are only part of the analytical process.

Successful incoming students understand that mathematical problem solving involves logical reasoning. Technology is important and relevant in understanding mathematics. However, students should be aware of the limitations of technology and recognize that calculators and computers are tools to assist but not replace the thinking process. Students should understand the basic mathematical terminology and use it appropriately. Students must pay attention to the wording of problems and move with ease between the symbolic representation of a problem and its verbal representation. Communication skills are as important in mathematics as in other disciplines. Students are expected to be able to read mathematics textbooks with understanding, write mathematical statements and expressions with clarity and cohesiveness, and present their understanding and findings in oral and visual formats.

Successful students will also present an orientation toward learning that presents itself as a willingness to work for significant periods of time on a single problem. Persistence is invaluable in the quest for a solution to a problem. Sustained inquiry – engaging in the process for more than a short period of time – is an important part of the process when solving a problem.

Oftentimes this process will help foster a deeper understanding, build confidence, and inspire learning.

Successful students demonstrate active participation in the process of learning mathematics by:

1. being willing to experiment with problems that have multiple solution methods;
2. demonstrating an understanding of the mathematical ideas behind the steps of a solution, as well as the solution;
3. showing an understanding of how to modify patterns to obtain different results;
4. showing an understanding of how to modify strategies to obtain different results; and
5. recognizing when a proposed solution does not work, analyzing why, and using the analysis to seek a valid solution.
6. demonstrating an ability to solve multi-step problems with a variety of strategies.

In the sections that follow, minimal entry-level competencies have been detailed in areas of numerical computation, algebra and geometry. Entry-level competencies related to data analysis, probability, and statistics are important in general as an educated person and in other disciplines. They are included in the cross disciplinary section of competencies. In the final section of mathematics entry-level competencies is a listing of the optimal competencies that would be expected for a student who plans to major in engineering, mathematics, physics, computer science, or another mathematically intensive area of study. [Note: These competencies are the same as those listed for Engineering Program.] Meeting these optimal competencies will allow a student to start the Calculus sequence in college.

I. Numerical Computation

Students must be proficient in numerical computation. We use the definition of proficiency from the National Mathematics Advisory Panel. "[T]he term *proficiency* means that students understand key concepts, achieve automaticity as appropriate, develop flexible, accurate, and automatic execution of the standard algorithms, and use these competencies to solve problems."[1] Conceptual understanding of these basic computations, although not explicitly stated, is assumed. While technology is useful in helping students explore and enhance their understanding of basic computations, their ability to conceptually understand and perform basic computations without the aid of technology increases the likelihood of success in college level mathematics courses.

Successful students are able to proficiently:

1. Apply mathematical operations to all real numbers in any form (including integers, rational numbers, radicals, and decimals), following the correct order of operations.
2. Calculate the sum, difference, product, and quotient of complex numbers and express the result in standard form.
3. Recognize and generate equivalent forms of fractions, decimals, and percents.

[1] National Mathematics Advisory Panel, *Foundations for Success*, U. S. Department of Education, March, 2008, p xvii.

4. Compare and order real numbers, including finding their approximate locations on the number line.
5. Apply laws of rational exponents to real number bases.
6. Recognize and generate equivalent representations (i.e., scientific notation) for very large and very small numbers, and perform mathematical operations on such numerical representations. Move flexibly between scientific notation and expanded form.
7. Compute quantities involving absolute value.
8. Apply the properties of real numbers (including commutative, associative, identity, inverse, and distributive properties).
9. Perform numerical computations involving units of measurement, standard and metric.
10. Communicate accurately using mathematical terminology (e.g., addend, sum, difference, factor, product, divisor, dividend, quotient, remainder, numerator, denominator, exponent, base, radicand, index and literal symbols including variables, parameters and constants).
11. Accurately record symbolic manipulations used in numerical computations, as well as the solutions of numerical computations (e.g., equal signs, inequality symbols, grouping symbols, exponents, subscripts, and solution sets).
12. Communicate accurately using set notation/terminology (e.g., set-builder notation, element of, well-defined, finite/infinite, subset, proper subset, $\varnothing$, cardinal number, equal, equivalent, and interval notation).
13. Estimate numerical computations and judge the reasonableness of the results of these computations.
14. Apply set operations and relations to sets (i.e., union, intersection, complement, and subsets).
15. Represent sets using graphic organizers, including Venn diagrams.

II. Algebra

Successful students are expected to bring a combination of hands-on skill and conceptual understanding of algebra.

1. Successful students know and apply basic algebraic concepts. They are able to:

    a. Add, subtract, multiply, and divide polynomials, rational expressions, and radical expressions.
    b. Divide polynomials.
    c. Apply properties of exponents and radicals.
    d. Factor polynomials (e.g., greatest common factor, grouping, trinomials, difference of squares, sum and difference of cubes).
    e. Simplify polynomials, rational expressions, and radical expressions.

2. Successful students use various appropriate techniques to solve basic equations and inequalities. They are able to:

    a. Solve linear equations and absolute value equations.

b. Solve linear inequalities and absolute value inequalities.
c. Solve systems of linear equations and inequalities with two variables, using algebraic or graphical methods.
d. Solve quadratic equations by factoring, completing the square, and using the quadratic formula.
e. Solve rational equations.
f. Solve radical equations.

3. Successful students distinguish among expressions, formulas, equations, functions and relations. They know when it is possible to simplify, solve, substitute or evaluate appropriately. In addition, they are able to:

   a. Use multiple representations for patterns and relationships.
   b. Correctly apply the algebraic language and notation for functions including domain and range.
   c. Compose and decompose functions and find inverses of basic functions.
   d. Identify and compare a variety of functions (e.g., constant, linear, quadratic, cubic, absolute value, piecewise, exponential and logarithmic functions) and apply the properties of each.

4. Successful students understand the relationship between equations and graphs. They are able to:

   a. Recognize basic forms of the equation of a line and graph the line without technology.
   b. Recognize the basic shape of the graph of a quadratic function; find the vertex; calculate and recognize the relationships among the solutions of the related quadratic equation, zeroes of the function and intercepts of the graph.
   c. Recognize and sketch the basic shapes of the graphs of the following functions: constant, linear, quadratic, cubic, square root, cube root, absolute value, exponential and logarithmic (without technology).
   d. Describe the effects of parameter changes on functions.
   e. Describe and sketch the effects of transformations on the graphs of functions.
   f. Represent data in a variety of ways (e.g. scatter plot, graph, and table) and select the most appropriate method.

5. Successful students understand algebra well enough to apply it procedurally and conceptually to a range of common problems. Successful students demonstrate the ability to work with formulas and symbols algebraically. They are able to:

   a. Recognize which type of function or expression best fits the context of a basic application.
   b. Use multiple representations to solve problems (e.g. analytic, numerical, and geometric).
   c. Represent algebraically and solve problems that include linear, quadratic, exponential, and logarithmic relationships.
   d. Use mathematics to solve applications from various fields (e.g. rates of change, compound interest, chemical mixture, population growth, and business).

e. Solve literal equations and formulas for a specified variable.
f. Communicate accurately using the vocabulary and symbols of algebra.

6. Successful students understand the appropriate use, as well as the limitation, of appropriate technology. They are able to:

a. Plot relevant graphs.
b. Use appropriate problem solving methods.
c. Recognize when the results produced are unreasonable or represent misinformation.

III. Geometry

Successful students must possess a basic body of knowledge including but not limited to the Pythagorean Theorem, formulas for perimeter, area, volume, and surface area. Successful students demonstrate an understanding of and can explain the mathematical ideas behind the steps of a solution as well as the solution. Successful students recognize when a proposed strategy does not work, analyze why, and use the analysis to seek a valid solution. Successful students understand the appropriate use as well as the limitations of technology.

Successful students are able to:

1. Apply properties of similarity and congruence.
2. Recognize and apply properties and theorems of parallel lines cut by a transversal.
3. Recognize and apply properties and theorems related to circles.
4. Determine the area and perimeter of plane figures and use the concept of conservation of area.
5. Apply the basic formulas for volume and surface area of solids.
6. Use deductive reasoning to develop and write simple geometric proofs.
7. Use inductive reasoning in problem situations to build a basis for the use of both proof and counter-examples.
8. Apply properties of similarity, particularly related to triangles, to find unknown geometric measurements including angle measurements, lengths of sides, areas, and volumes.
9. Recognize and represent solids and surfaces in three-dimensional space from a two-dimensional representation (e.g. recognize the features of a three-dimensional object – faces, edges, vertices, and shape).
10. Use coordinate geometry to make connections between algebra and geometry.
    a. Describe lines in the coordinate plane using slope-intercept and point-slope form.
    b. Use slopes to describe the steepness and direction of lines in the coordinate plane and to determine if lines are parallel, perpendicular, or neither.
    c. Relate geometric and algebraic representations of lines, segments, simple curves, and circles.
    d. Derive and use the formula for distance between two points.
11. Apply the definitions of sine, cosine, and tangent using right triangle trigonometry.
12. Describe and represent transformations and symmetries of plane figures.

13. Make connections between analytic, numerical, and geometric methods to solve problems.

IV. Optimal Competencies to Start the Calculus Sequence.

Successful students will demonstrate proficiency in the entry-level competencies listed above in the areas of numerical computation, algebra, and geometry. Successful students will also bring strong conceptual understanding and demonstrated proficiency in the following areas. All the calculations are able to be performed without the aid of technology.

1. Algebra and Real Numbers – Successful students are able to:

    a. Use symbols and operators to represent ideas and objects and the relationships existing between them.
    b. Express the relationship between measures of the physical world and be able to convert units in a given problem (e.g. velocity, distance, time, etc.).
    c. Know and apply the following properties of the real number system: identity, associative, commutative, inverses, and distributive.
    d. Express numbers using scientific notation.
    e. Write a number as the product of factors.

2. Radicals and Exponents – Successful students are able to:
    a. Convert between radical and rational exponent form.
    b. Manipulate algebraic expressions that contain integer and rational exponents.
    c. Simplify expressions containing radicals or rational exponents.

3. Algebraic Expressions – Successful students are able to:
    a. Add, subtract, multiply and divide algebraic expressions.
    b. Simplify algebraic expressions.

4. Linear Equations, Inequalities and Absolute Values – Successful students are able to:
    a. Understand the meaning of solutions to linear and rational equations and be able to solve such equations whenever appropriate.
    b. Determine the equation of a line.
    c. Determine the equation of a line that is parallel or perpendicular to a given line.
    d. Solve a two variable system of linear equations by substitution and elimination.
    e. Solve linear equations and inequalities [graphically and algebraically].
    f. Understand the meaning of solutions to linear and absolute value inequalities. Solve linear equations and inequalities with absolute values.
    g. Understand using matrix solutions to linear systems of equations in more than two variables and be able to use effective ways to find and express possible solutions.
    h. (Recommended) Understand the concepts of matrices and their inverses (if exist), matrix operations, determinants, and be able to perform these computations. Understand how matrices are used to model and solve systems of linear equations and be able to perform required appropriate computations.

5. Polynomials, Zeros of Polynomials, and Rational Inequalities – Successful students are able to:

   a. Understand the properties and graphs of polynomial functions.
   b. Understand the zero factor or zero product property; understand the meaning of zeros of polynomial functions and their connection to the graphs of these functions.
   c. Solve for the roots of a polynomial by factoring.
   d. Understand the meaning of the Remainder Theorem and its application in evaluating polynomial functions.
   e. Understand the meaning of the Factor Theorem and its application in solving polynomial equations.
   f. Understand the meaning of solutions to polynomial~~,~~ and rational inequalities and be able to solve such inequalities whenever appropriate.
   g. Solve simple polynomial inequalities.
   h. Solve simple rational inequalities.
   i. Understand the importance of the Fundamental Theorem of Algebra, its application to polynomial equations and its connection to complex numbers.

6. Functions, Graphs and Graphing – Successful students are able to:

   a. Identify the independent and dependent variables of a function.
   b. Evaluate a function at a value.
   c. Determine the domain and range of a real valued function.
   d. Understand the concept of combining functions arithmetically and by composition and be able to perform these operations and recognize the resulting functions and their properties.
   e. Evaluate the composition of functions.
   f. Understand the concept of piecewise-defined functions and be able to translate this knowledge to their properties and graphs.
   g. Graph equations and inequalities.
   h. Understand the concept of transformations (e.g. shifting, reflecting, stretching, and shrinking) of functions and be able to recognize and apply such knowledge when graphing functions.
   i. Transform the graph of a known function.
   j. Determine whether a basic algebraic function is invertible and, if so, be able to calculate the function's inverse. Know the relationships between a function and its inverse.
   k. Understand the properties and graphs of rational functions and be able to generate appropriate information, including axes, intercepts, intervals of continuity, asymptotes (horizontal, vertical, and oblique), and roots. Be able to graph a rational function showing its salient characteristics without using a calculator using properties of a rational function.
   l. Know the general characteristics and shapes of the graphs of polynomial, simple rational, logarithm, exponential and trigonometric functions.
   m. Understand the properties and graphs of parabolas, ellipses, and/or hyperbolas and be able to perform basic related algebraic/graphing operations.

7. Equations of Quadratic Type and Complex Numbers – Successful students are able to:

   a. Understand the concept of complex numbers and be able to perform operations involving them.
   b. Calculate the sum, difference, product, and quotient of complex numbers and express the result in standard form.
   c. Understand the process of completing the square of a quadratic expression and its connection to solving quadratic equations and graphing.
   d. Solve for real and complex roots using the quadratic formula.
   e. Solve a system of quadratic equations in two variables by substitution.
   f. Understand the relationship between quadratic functions and parabolas, and be able to connect such knowledge to quadratic equations.

8. Logarithmic and Exponential Functions – Successful students are able to:

   a. Understand the meaning of solutions to exponential and logarithmic equations, and be able to apply the inverse relationship between exponentials and logarithms to equations involving them whenever appropriate.
   b. Apply the properties of logarithms and the relationship to exponentials. Be able to perform operations on logarithms.
   c. Know the properties of logarithmic and exponential~~s~~ functions and use them to simplify logarithmic expressions.
   d. Know how to solve simple logarithmic and exponential equations.
   e. Understand the properties and graphs of logarithmic and exponential functions and be able to evaluate and graph such functions.
   f. Understand the meaning of exponential growth and decay and apply the knowledge of exponential and logarithmic functions to model applications.

9. Analytic Geometry – Successful students are able to:

   a. Know and apply the distance formula between two points.
   b. Understand the geometric concepts of angle (e.g. initial side, terminal side, coterminal angles, degree, radian, central angle, circular arc length, circular sector area, and reference angle) and be able to apply appropriate properties.
   c. Know and apply the circumference/perimeter and area formulas for circles, triangles and rectangles.
   d. Know and apply the surface and volume formulas for cylinders, spheres and rectangular solids.
   e. Know the relationship~~s~~ between similar triangles.
   f. Know and apply the Pythagorean Theorem to simple geometric problems.

10. Use of Mathematics to Solve Applications from Various Fields – Successful students are able to:

a. Apply the acquired understanding and knowledge of functions to model appropriate real-world situations and draw mathematical conclusions.
b. Understand the underlying principle of variation and how it is used to model many applications.
c. (Recommended) Understand the meaning of solutions of systems of nonlinear equations and use effective ways to find and express possible solutions.
d. Understand the meaning of compound interest and apply the knowledge of exponential functions to model this application.
e. (Recommended) Be able to use trigonometry to model and solve basic applied problems.

11. Trigonometric Functions and Their Inverses – Successful students are able to:

a. Define each of the six trigonometric functions ($\cos\theta$, $\sin\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$) in terms of the sides of a right triangle and in terms of coordinates on a unit circle.
b. Define each of the six trigonometric functions in terms of sin(θ) and cos(θ).
c. Understand the concepts of the six trigonometric functions, both in terms of a unit circle and a right triangle, and be able to apply such knowledge.
d. Know the domain and ranges for the sine, cosine and tangent functions; know why domains of inverse trigonometric functions have the usual restrictions.
e. Convert angle measures between degrees and radians.
f. Memorize and use the 30/60/90 and 45/45/90 degree reference triangles.
g. Understand the graphs of the six trigonometric functions and be able to recognize and apply such knowledge (including incorporation of appropriate transformations: shifting, reflecting, stretching, and shrinking).

12. Trigonometric Identities and Equations − Successful students are able to:

a. Understand the general nature of proving trigonometric identities and be able to perform such task appropriately.
b. Know and apply the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$.
c. Understand the general nature of trigonometric equations and be able to solve such equations whenever appropriate.
d. Be familiar with useful formulas (e.g. addition and subtraction, double-angle, half-angle, product-to-sum, sum-to-product, law of sines, law of cosines, and Heron's).
e. Understand the concepts and graphs of inverse trigonometric functions and their related properties and be able to perform appropriate operations.
f. Understand the trigonometric form and its geometric interpretation for complex numbers, and be able to recognize and perform conversions.
g. Understand the multiplication and division of complex numbers in trigonometric form and their respective geometric interpretations.
h. Understand De Moivre's Theorem and its geometric interpretation, and be able to apply the concept to find roots of complex numbers.

i. Understand the basic concepts and operations of two-dimensional vectors, their respective geometric interpretation, and the trigonometric aspect of the inner (dot) product.
j. Understand the geometry of complex numbers.